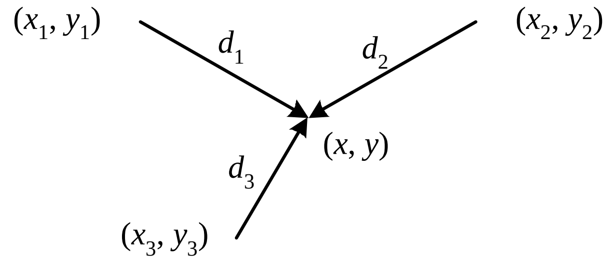
Solution for Magic Trig Cache Algebra.

The geometry of the problem is shown in Fig. 1. We wish to calculate the unknown coordinate (x, y) from three known coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3).



The algebraic solution is clever, but straightforward. Recall from Pythagorus' theorem:

$$(x - x_1)^2 + (y - y_1)^2 = d_1^2 .$$

Expanding, you get:

$$x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 = d_1^2 .$$

Grouping all constants on the same side of the equation:

$$x^2 - 2xx_1 + y^2 - 2yy_1 = d_1^2 - x_1^2 - y_1^2 = 2C_1 .$$

I know the factor of 2 seems arbitrary, but it helps later. Likewise,

$$x^2 - 2xx_2 + y^2 - 2yy_2 = d_1^2 - x_2^2 - y_2^2 = 2C_2 \quad \text{and}$$

$$x^2 - 2xx_3 + y^2 - 2yy_3 = d_3^2 - x_3^2 - y_3^2 = 2C_3 .$$

Subtract the first two equations from each other, and you get:

$$2x(x_2 - x_1) + 2y(y_2 - y_1) = 2(C_1 - C_2) \Rightarrow x = \frac{y(y_1 - y_2) + (C_1 - C_2)}{(x_2 - x_1)} .$$

This equation defines a line on which both intersections of the circles centered at the points must lie. If you do the same to the second and third equations, but solve for y, you get:

$$2x(x_3 - x_2) + 2y(y_3 - y_2) = 2(C_2 - C_3) \Rightarrow y = \frac{x(x_2 - x_3) + (C_2 - C_3)}{(y_3 - y_2)} .$$

Multiply this last equation by $\dfrac{(x_2 - x_1)}{2(x_3 - x_2)}$ and subtract from the previous:

$$y\left[(y_2 - y_1) - \frac{(x_2 - x_1)}{(x_3 - x_2)}(y_3 - y_2)\right] = (C_1 - C_2) - \frac{(C_2 - C_3)(x_2 - x_1)}{(x_3 - x_2)} .$$

Straightening up, you get the final answer:

$$y = \frac{(x_3 - x_2)(C_1 - C_2) - (x_2 - x_1)(C_2 - C_3)}{(x_3 - x_2)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_2)} . \qquad \text{Voila!}$$